Journal of Differential Equations **176**, 356–386 (2001)
doi:10.1006/jdeq.2000.3987, available online at http://www.idealibrary.com on IDEAL®

Optimal Stability for the Inverse Problem of Multiple Cavities[1]

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We deal with the determination of finitely many cavities in a planar inhomogeneous conductor from one current and voltage measurement collected on the exterior boundary. We prove stability estimates under essentially minimal a priori regularity assumptions. We construct an explicit example showing the optimality of such stability estimates.

1. INTRODUCTION

Consider a simply connected bounded open set Ω of the plane and a closed subset Σ which is the union of finitely many, pairwise disjoint, closed simply connected subset σ_i, $i=1, \ldots, n$, each of them coinciding with the closure of its interior (that is for any $i=1, \ldots, n$, $\sigma_i = \overline{\mathring{\sigma}_i}$ where $\mathring{\sigma}_i$ denotes the interior part of σ_i).

The Neumann problem

$$
\begin{aligned}
&\operatorname{div}(A\,\nabla u)=0 && \text{in} \quad \Omega \backslash \Sigma,\\
&A\,\nabla u \cdot \nu = 0 && \text{on} \quad \partial\sigma_i,\ i=1, \ldots, n, \qquad (1.1)\\
&A\,\nabla u \cdot \nu = \psi && \text{on } \partial\Omega
\end{aligned}
$$

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0022-0396/01 $35.00

provides a model for the electrostatic potential u in the conductor Ω when each σ_i, $i=1, \dots, n$, represents a cavity inside it, Σ being therefore a *multiple cavity*, ψ is the applied current density and A is the, possibly anisotropic and inhomogeneous, conductivity tensor. Here, we assume: $\int_{\partial\Omega} \psi = 0$, $\psi \not\equiv 0$, $A \in L^\infty(\Omega)$ is uniformly elliptic, and we denote by v the exterior normal to $\Omega \setminus \Sigma$.

We shall deal with the inverse problem of determining the multiple cavity Σ, when, given Ω, A and ψ, the potential u is measured on an open portion Γ of the exterior boundary $\partial\Omega$.

Such a problem presents some similarities with other well-known inverse boundary value problems.

(I) The so-called inverse problem of cracks is the one in which each component of Σ, instead of having interior points, is just a simple arc. For this case, it is well known that, either when Σ is assumed to have only one component, [15], or finitely many ones, [6], two distinct, suitably chosen measurements are sufficient and necessary to uniquely determine the multiple crack. Moreover, stability estimates for the determination of a single crack have been obtained. See [8] for an updated account on such results.

(II) Consider for simplicity $A \equiv I$, then (1.1) can be viewed as the limit as $k \to 0$ of the problems

$$(1.1_k) \qquad \begin{aligned} &\operatorname{div}((1+(k-1)\,\chi_\Sigma)\,\nabla u_k) = 0 && \text{in } \Omega, \\ &\nabla u_k \cdot v = \psi && \text{on } \partial\Omega. \end{aligned}$$

Here χ_Σ is the characteristic function of Σ. In this case Σ represents an inclusion in Ω, whose conductivity gets smaller as $k \to 0$. When $k \neq 1$ is fixed, the relative inverse problem of determining Σ is known as the inverse conductivity problem with one measurement. Plenty of papers have been devoted to this problem but, still, the uniqueness question remains open. For references, see, for instance, [7].

Contrary to the above stated inverse problems, in the case of cavities the uniqueness with a single measurement is nearly straightforward. Let us outline a proof, suited to the two-dimensional setting, which has the advantage of requiring very little about the regularity of the conductivity A and of the boundaries of the conductor Ω and of the cavities. Let v be a *stream function* associated to u, (a notion that generalizes the one of harmonic conjugate), namely a function satisfying

$$(1.2) \qquad \nabla v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} A\, \nabla u \qquad \text{almost everywhere in } \Omega \setminus \sigma.$$

It can be seen that, due to (1.1), such a function exists, it is single valued, and it satisfies, for some unknown constants c_i, $i = 1, \ldots, n$,

$$\begin{aligned} &\operatorname{div}(B\,\nabla v) = 0 && \text{in} \quad \Omega \setminus \Sigma, \\ &v = c_i && \text{on} \quad \partial\sigma_i, \, i = 1, \ldots, n, \\ &v = \Psi && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

and also, in a weak sense,

$$\int_\beta B\,\nabla v \cdot \nu = 0 \qquad \text{for every smooth Jordan curve } \beta \subset \Omega \setminus \Sigma. \tag{1.4}$$

Here $B = (\det A)^{-1} A^T$, $(\cdot)^T$ denoting transpose, and Ψ is an antiderivative of ψ along $\partial\Omega$. See, for details, [6] and also [8].

Notice also that v can be continuously extended to Ω by setting $v|_{\sigma_i} = v|_{\partial\sigma_i} = c_i$ for any $i = 1, \ldots, n$.

Suppose now Σ' is another multiple cavity and let u', v' be the solution to (1.1), (1.3) respectively when Σ is replaced with Σ'. If Σ, Σ' give rise to the same boundary measurement, that is $u|_\Gamma = u'|_\Gamma$, then v, v' have the same Cauchy data on Γ. By the unique continuation property, and the maximum principle, one obtains $v \equiv v'$ in Ω (see again [6] for details).

Hence, if we had $\Sigma' \setminus \Sigma \neq \emptyset$ then Σ' would have some of its interior points inside $\Omega \setminus \Sigma$ and we would obtain $v = v' \equiv \text{const.}$ on an open subset of $\Omega \setminus \Sigma$. Again by unique continuation, we obtain $v \equiv \text{const.}$ which is impossible since $\psi \not\equiv 0$.

The aim of this paper is to prove stability estimates for the inverse problem of cavities under very general assumptions on the conductivity, on the regularity of the boundaries of Ω and of the cavities and on the prescribed current density ψ. We shall prove our stability results under different type of regularity assumptions on Σ, see Theorem 2.1. Moreover we shall show their optimality by an explicit example, Theorem 4.1. See also [5], where an example, different in various respects, but of the same nature, was presented for the so-called material loss (or corrosion) problem.

Our approach has some common features with the one used in [8] for the stability estimates in the determination of a single crack from two measurements and it will require a sequence of intermediate steps: first of all we shall prove an inverse Hölder estimate on the function $f = u + iv$, see Theorem 3.3. Then, according to the different a priori regularity assumptions on Σ, we shall derive stability estimates for a Cauchy type problem, see Proposition 3.5 and 3.6, which, along with the inverse Hölder estimate previously recalled, will allow us to conclude the proof of Theorem 2.1.

However, there are various relevant additional difficulties. First, the possible presence of multiple cavities introduces technical complications in the treatment of quasiconformal mappings between multiply connected domains, a crucial step in this treatment being Lemma 3.2 which provides estimates on the size deformation of a circular domain (that is a multiply connected domain bounded by finitely many circles) under the effect of a k-quasiconformal mapping.

Second, we recall that in the stability estimates obtained in [4], [8] for the crack problem, and, more specifically, in [20], for the problem of a cavity, the prescribed Neumann data ψ was assumed to satisfy certain conditions on its sign changes which enabled to show that, in a generalized sense, the corresponding potential u had no interior critical points. Here no such assumption on ψ will be made, in fact any nontrivial data ψ will suit our purpose. On the other hand, we shall obtain, as it is reasonably expected, that the constants in the stability estimates depend on the oscillation character of ψ. That is, the less is the oscillation of ψ the better is the stability. Roughly speaking, such an oscillation character will be controlled by the quantity H_2, appearing in (2.6) below, which dominates a ratio of two different norms for ψ. We shall prove that, under such a bound on the oscillation of ψ, taking $f=u+iv$ where v is the above mentioned stream function associated to u, and fixing any $z^0 \in \overline{\Omega\setminus\Sigma}$ then, locally, $|f-f(z^0)|$ can be dominated from below by an explicit function vanishing at finitely many points and with finite order (see Theorem 3.3). We believe that such type of estimate, which to the authors' knowledge is new in the context of elliptic equations in divergence form and measurable coefficients in two variables, may prove to be useful also for other purposes and especially for other inverse boundary value problems.

We wish to mention that stability estimates for a strictly related problem of determination of an interior boundary have been obtained in [13], where they consider the case of a single cavity σ, they assume the conductivity A to be homogeneous, $A\equiv I$, and the regularity assumptions on the boundaries are slightly different.

Let us also mention that a companion problem, in a different topological setting, when $\sigma\subset\bar{\Omega}$ intersects $\partial\Omega$ on a nontrivial arc in such a way that $\Omega\setminus\sigma$ remains simply connected, has already been treated by various authors, [5], [9], [10], [12, 14], [17] and [20, 21].

An extended account on these and other related topics can be found in the doctoral dissertation [22] of which the present research constitutes a part.

The plan of the paper is the following. In Section 2, we present our basic assumptions and state our main Theorem 2.1, which contains our stability results under various type of a priori assumptions on Σ. Section 3 contains the proof of Theorem 2.1 in its various steps. Finally, in Section 4, we illustrate an example showing the optimality of the stability estimates (II) and (III) in Theorem 2.1. In fact such an example provides a much stronger statement, Theorem 4.1, showing that logarithmic stability is the best possible also when all pairs of boundary measurements $\{u|_{\partial\Omega}, A\,\nabla u\cdot\nu|_{\partial\Omega}\}$ are available.

2. THE STABILITY THEOREM

Given $z \in \mathbb{C}$ and $r > 0$, we denote by $B_r(z)$ the open disc with center z and radius r and by $B_r[z]$ its closure, that is $B_r[z] = \overline{B_r(z)}$.

We shall need, in several places, quantitative notions of smoothness for the boundary of Ω and the boundaries of the cavities. Such assumptions can be summarized as follows.

Given an integer $k = 0, 1, 2, \ldots$, a number α, $0 < \alpha \leqslant 1$, and a finite family of simple closed curves γ_i, $i = 1, \ldots, n$, such that the domains bounded by each γ_i are pairwise disjoint, we shall say that this family is $C^{k,\alpha}$ with constants δ, $M > 0$ if for any $z \in \bigcup_{i=1}^n \gamma_i$, $(\bigcup_{i=1}^n \gamma_i) \cap B_\delta(z)$ is given, up to a rigid transformation, by the graph $\{y = \phi(x), x^2 + y^2 < \delta^2\}$ of a $C^{k,\alpha}$ function ϕ on $[-\delta, \delta]$ such that $\|\phi\|_{C^{k,\alpha}[-\delta,\delta]} \leqslant M$.

We shall especially focus on the case $k = 0$, $\alpha = 1$, in which case we shall speak of Lipschitz curves.

Use will be also made of the following notion. Given two finite families of simple closed curves, γ_i, $i = 1, \ldots, n$, and γ'_j, $j = 1, \ldots, m$, both satisfying the assumption that the domains bounded by each of the curves of the same family are pairwise disjoint, we shall say that they are *Relative Lipschitz Graphs* (RLG for short) with constants δ, M if for every $z \in (\bigcup_{i=1}^n \gamma_i) \cup (\bigcup_{j=1}^m \gamma'_j)$, there exists a coordinate system (x, y) with origin in z such that with respect to these coordinates $(\bigcup_{i=1}^n \gamma_i) \cap B_\delta(z) = \{y = \phi(x), x^2 + y^2 < \delta^2\}$ and $(\bigcup_{j=1}^m \gamma'_j) \cap B_\delta(z) = \{y = \phi'(x), x^2 + y^2 < \delta^2\}$ where ϕ and ϕ' are Lipschitz on $[-\delta, \delta]$ with Lipschitz norm bounded by M. Moreover we assume that $\{y < \phi(x), x^2 + y^2 < \delta^2\}$ and $\{y < \phi'(x), x^2 + y^2 < \delta^2\}$ are contained in one of the domains bounded by γ_i, $i = 1, \ldots, n$, and γ'_j, $j = 1, \ldots, m$, respectively. We remark that either $(\bigcup_{i=1}^n \gamma_i) \cap B_\delta(z)$ or $(\bigcup_{j=1}^m \gamma'_j) \cap B_\delta(z)$ might be empty. In this case it is enough to have ϕ (or respectively ϕ') larger than or equal to δ, if $B_\delta(z)$ is contained in one of the domains bounded by a curve belonging to the first (or respectively to the second) family, or otherwise $\phi \leqslant -\delta$ ($\phi' \leqslant -\delta$ respectively).

Before stating our main Theorem, let us illustrate the main a priori assumptions.

Prior Information on the Domain

Given positive constants δ, M and L, let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ whose boundary $\partial\Omega$ is a simple, closed Lipschitz curve with constants δ, M and length bounded by L.

Let us observe that by the a priori information we may deduce that the length of $\partial\Omega$ is greater than or equal to δ and there exists a positive constant M_1 depending on δ, M and L only such that

$$\text{length}_{\partial\Omega}(z_0, z_1) \leqslant M_1\,|z_0 - z_1|, \qquad \text{for every} \quad z_0, z_1 \in \partial\Omega, \tag{2.1}$$

where $\text{length}_{\partial\Omega}(z_0, z_1)$ denotes the length of the smallest arc in $\partial\Omega$ connecting z_0 to z_1.

The bound on the length of $\partial\Omega$ allows us to obtain an upper bound on the diameter, and consequently on the measure, of Ω by a constant depending on L only. Finally, the measure of Ω can be bounded from below by a positive constant depending on δ and M only.

Prior Information on the Conductivity

Given λ, $\Lambda > 0$, let $A = A(z)$, $z \in \Omega$, be a 2×2 matrix with bounded measurable entries verifying

$$\begin{aligned} &\text{(a)} \quad A(z)\,\xi\cdot\xi \geqslant \lambda\,|\xi|^2 \text{ for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } z \in \Omega; \\ &\text{(b)} \quad |a_{ij}(z)| \leqslant \Lambda \text{ for every } i, j = 1, 2 \text{ and for a.e. } z \in \Omega. \end{aligned} \tag{2.2}$$

Prior Information on the Multiple Cavity

We shall assume that $\Sigma \subset \Omega$ is the union of finitely many, pairwise disjoint, closed and not empty sets σ_i, $i = 1, \ldots, n$, $n \geqslant 1$, each of them bounded by a simple closed curve γ_i. Concerning the regularity of the curves γ_i, we shall pose various alternative assumptions in the statement of Theorem 2.1.

Moreover we shall assume

$$\text{dist}(z, \partial\Omega) \geqslant \delta \qquad \text{for any} \quad z \in \Sigma. \tag{2.3}$$

We remark that this kind of definition guarantees that $\Omega\backslash\Sigma$ is a connected open set.

Prior Information on the Boundary Data

The current density on the boundary will be given by a non trivial function $\psi \in L^2(\partial\Omega)$ with zero mean, that is $\int_{\partial\Omega}\psi = 0$.

We define the antiderivative along $\partial\Omega$ of ψ as

$$\Psi(s) = \int \psi(s)\,\mathrm{d}s, \tag{2.4}$$

where the indefinite integral is taken with respect to the arclength on $\partial\Omega$ oriented in the counterclockwise direction.

We recall that the function Ψ is defined up to an additive constant. For the time being, we normalize Ψ in such a way that $\int_{\partial\Omega} \Psi = 0$ and for this choice of the additive constant we prescribe that, for given constants H, $H_1 > 0$, we have

$$\text{(2.5)} \qquad \begin{aligned} &\text{(a)} \quad \|\psi\|_{L^2(\partial\Omega)} \leqslant H; \\ &\text{(b)} \quad \|\Psi\|_{L^2(\partial\Omega)} \geqslant H_1. \end{aligned}$$

From (2.5)(a) and (2.5)(b) we immediately infer

$$\text{(2.6)} \qquad \frac{\|\psi\|_{L^2(\partial\Omega)}}{\|\Psi\|_{L^2(\partial\Omega)}} \leqslant H_2,$$

where $H_2 = H/H_1$ and Ψ has zero average.

Furthermore, by (2.5)(a) and (2.1), Ψ verifies for any $z_0, z_1 \in \partial\Omega$

$$\text{(2.7)} \qquad |\Psi(z_0) - \Psi(z_1)| \leqslant H(\text{length}_{\partial\Omega}(z_0, z_1))^{1/2} \leqslant H_3\, |z_0 - z_1|^{1/2},$$

where $H_3 = HM_1^{1/2}$.

Prior Information on the Measurements

Let $\Gamma \subset \partial\Omega$ be a subarc whose length is greater than δ.

The set of constants $\{\delta, M, L, \lambda, \Lambda, H, H_1\}$ will be referred to as the *a priori data*.

Let us finally recall that, under the stated assumptions, a weak solution to (1.1), that is a function $u \in W^{1,2}(\Omega \backslash \Sigma)$ satisfying

$$\text{(2.8)} \qquad \int_{\Omega\backslash\Sigma} A\, \nabla u \cdot \nabla\varphi = \int_{\partial\Omega} \psi\varphi \qquad \text{for every} \quad \varphi \in W^{1,2}(\Omega\backslash\Sigma),$$

exists and it is unique up to an additive constant.

Given another multiple cavity Σ', satisfying the a priori assumptions, with components σ'_j, $j = 1, ..., m$, $m \geqslant 1$, whose boundaries are simple closed curves denoted by γ'_j, $j = 1, ..., m$, we shall denote by u' a solution to (2.8) when Σ is replaced with Σ'.

THEOREM 2.1. *Let the above prior assumptions be satisfied. Suppose*

$$\text{(2.9)} \qquad \|u - u'\|_{L^\infty(\Gamma)} \leqslant \varepsilon.$$

We have:

(I) *If the two families of boundaries* γ_i *and* γ'_j *are Lipschitz with constant* δ, M, *then*

$$\mathrm{d}_H(\Sigma, \Sigma') \leqslant \omega(\varepsilon), \tag{2.10}$$

where ω: $(0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\omega(\varepsilon) \leqslant K(\log|\log \varepsilon|)^{-\beta} \qquad \textit{for every } \varepsilon, \quad 0 < \varepsilon < 1/\mathrm{e} \tag{2.11}$$

and K, $\beta > 0$ *depend on the* a priori data *only.*

Furthermore there exists a constant $\varepsilon_0 > 0$, *depending on the* a priori data *only, so that if* $\varepsilon \leqslant \varepsilon_0$ *then the number of connected components of* Σ *and* Σ' *is the same, for instance equal to* n, *and, up to rearranging their order, we have*

$$\mathrm{d}_H(\sigma_i, \sigma'_i) \leqslant \omega(\varepsilon), \qquad \textit{for every} \quad i = 1, \ldots, n, \tag{2.12}$$

ω *as in* (2.11).

(II) *If* γ_i *and* γ'_j *are RLG with constants* δ, M, *then* (2.10) *holds where in this case* ω: $(0, +\infty) \mapsto (0, +\infty)$ *satisfies*

$$\omega(\varepsilon) \leqslant K_1 \, |\log \varepsilon|^{-\beta_1} \qquad \textit{for every } \varepsilon, \quad 0 < \varepsilon < 1/\mathrm{e} \tag{2.13}$$

and K_1, $\beta_1 > 0$ *depend on the* a priori data *only.*

Also in this case, if $\varepsilon \leqslant \varepsilon_0$, $\varepsilon_0 > 0$ *depending on the* a priori data *only,* Σ *and* Σ' *have the same number* n *of connected components, and, again after rearranging their order,* (2.12) *holds with* ω *as in* (2.13).

(III) *If, for some* $k = 1, 2, \ldots$ *and some* α, $0 < \alpha \leqslant 1$, γ_i *and* γ'_j *are* $C^{k,\alpha}$ *with constants* δ, M *then* Σ *and* Σ' *verify* (2.10) *where* ω *is as above in* (2.13) *with* K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only.*

As before, we may find $\varepsilon_0 > 0$ *depending on the* a priori data, *on* k *and on* α *only, such that if* $\varepsilon \leqslant \varepsilon_0$ *both* Σ *and* Σ' *have* n *connected components, which ordered in a suitable way verify* (2.12) *with* ω *as in* (2.13), K_1, $\beta_1 > 0$ *depending on the* a priori data *and on* k *and* α *only. Moreover, for any* $i = 1, \ldots, n$, *there exist regular parametrisations* $z_i = z_i(t)$ *and* $z'_i = z'_i(t)$, $0 \leqslant t \leqslant 1$, *of* γ_i *and* γ'_i *respectively such that for every* $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$,

$$\|z_i - z'_i\|_{C^{k,\tilde{\alpha}}[0,1]} \leqslant K_2 \omega(\varepsilon)^{(\alpha - \tilde{\alpha})/(k+\alpha)}, \tag{2.14}$$

where ω still verifies (2.13) *and K_2 depends on the* a priori data*, on k, on α and on $\tilde{\alpha}$ only.*

First, we recall that $d_H(\cdot, \cdot)$ denotes the Hausdorff distance between bounded closed sets.

Next, we observe that the assumption made at point (II) can be viewed as a non-trivial closeness condition between Lipschitz curves. In fact there are examples of pairs of Lipschitz curves which are arbitrarily close in the sense of the Hausdorff distance but are not RLG, see [21].

The key step of (III) will indeed be the following. If Σ and Σ' are a priori known to be $C^{k,\alpha}$, $k \geqslant 1$, $\alpha > 0$, with given constants δ, M and they are sufficiently closed in the Hausdorff sense then they are RLG.

3. PROOF OF THEOREM 2.1

For the time being, we shall assume that Σ and Σ' satisfy the assumptions stated in (I) of Theorem 2.1. It is easy to observe that if Σ and Σ' verify the assumptions (II) or (III) of Theorem 2.1, then they verify also (I) of the same Theorem. In view of assumption (I), let us remark some properties of Σ. The same properties are clearly shared also by Σ'. We have that the boundary γ_i of any of the components σ_i of Σ has a length bounded by a constant depending on the *a priori data* only. Furthermore there exist a positive constant δ_1 and an integer N, depending on the *a priori data* only, such that

$$\operatorname{dist}(\sigma_i, \sigma_j) \geqslant \delta_1, \qquad \text{for every} \quad i \neq j, \tag{3.1}$$

and

$$n = \text{number of connected components of } \Sigma \leqslant N. \tag{3.2}$$

In the Introduction we have considered, (1.2), the notion of stream function and we have stated that there exists a single valued stream function v associated to u, u weak solution to (1.1). Let us recall that v satisfies the Dirichlet type boundary value problem (1.3) with condition (1.4), where the constants c_i are unknown, $B = (\det A)^{-1} A^T$ and Ψ is defined as in (2.4). We shall always assume that v is extended to Ω by setting $v|_{\sigma_i} = v|_{\partial\sigma_i} = c_i$ for any $i = 1, \ldots, n$.

Then the complex valued function $f = u + iv$, defined in $\Omega \backslash \Sigma$, satisfies the following first order Beltrami type equation

$$f_{\bar{z}} = \mu f_z + \nu \overline{f_z} \qquad \text{almost everywhere in} \quad \Omega \backslash \Sigma, \tag{3.3}$$

where μ and v are bounded measurable, complex valued coefficients, satisfying

$$|\mu|+|v| \leqslant k<1 \qquad \text{almost everywhere in} \quad \Omega \backslash \Sigma \tag{3.4}$$

with k depending on λ, Λ only.

For any k, $0 \leqslant k<1$, we say that a function f is a *k-quasiconformal function* in a domain D if it satisfies (3.3), (3.4). A univalent solution to (3.3), (3.4) is said a *k-quasiconformal mapping*. A function f is a *quasiconformal function*, respectively *mapping*, if it is a *k-quasiconformal function*, respectively *mapping*, for some k, $0 \leqslant k<1$. Concerning quasiconformal functions, their properties and characterizations we refer to [18].

A *circular domain D* will be, by definition, a bounded domain whose boundary is composed by a finite number of circles, that is $D=B_R(z) \backslash \bigcup_{i=1}^n B_{r_i}[z_i]$, where n is a positive integer, for any $i=1, \ldots, n$ $B_{r_i}[z_i] \subset B_R(z)$ and the cavities $B_{r_i}[z_i]$ are pairwise disjoint. We call $\partial B_R(z)$ the *exterior boundary* and $\bigcup_{i=1}^n B_{r_i}[z_i]$ the *multiple cavity* of the circular domain D. Furthermore we introduce the following notations. For any cavity $B_{r_i}[z_i]$, $i=1, \ldots, n$, let us denote

$$d_i=\operatorname{dist}\left(B_{r_i}[z_i], \bigcup_{j \neq i} B_{r_j}[z_j] \cup \partial B_R(z)\right).$$

We shall say *minimal radius* (*of the multiple cavity*) the number $\min \{r_i \mid i=1, \ldots, n\}$ and *separation distance* (*of the multiple cavity*) the number $\min \{d_i \mid i=1, \ldots, n\}$.

PROPOSITION 3.1. *Under the assumptions of Part* (I) *of Theorem* 2.1, *let u be a weak solution to* (1.1) *and v be its stream function, solution to* (1.3) *with condition* (1.4). *Then the following representation holds*

$$f=F \circ \chi, \tag{3.5}$$

where χ: $\Omega \backslash \Sigma \mapsto D$ is a quasiconformal mapping satisfying

$$|\chi(x)-\chi(y)| \leqslant C_1\, |x-y|^{\alpha_1} \qquad \textit{for any} \quad x, y \in \Omega \backslash \Sigma \tag{3.6}$$

and

$$|\chi^{-1}(x)-\chi^{-1}(y)| \leqslant C_1\, |x-y|^{\alpha_1} \qquad \textit{for any} \quad x, y \in D, \tag{3.7}$$

$D = B_1(0) \setminus \bigcup_{i=1}^{n} B_{r_i}[z_i]$ is a circular domain such that its exterior boundary is $\partial B_1(0)$ and is the image through χ of $\partial\Omega$ and the minimal radius and the separation distance of its multiple cavity are greater than $\delta_2 > 0$ and $F = U + iV$ is a holomorphic function on D. Here $C_1 > 0$, α_1, $0 < \alpha_1 < 1$, and $\delta_2 > 0$ depend on the a priori data *only.*

Proof. We may find a bi-Lipschitz transformation χ_1 from $\mathbb{C}$ onto itself such that the image through χ_1 of $\Omega \setminus \Sigma$ is a circular domain $\tilde{D}$ such that $0 \in \tilde{D}$, its exterior boundary $\partial B_1(0) = \chi_1(\partial\Omega)$ and the minimal radius and separation distance of its multiple cavity are greater than $\delta_3 > 0$, δ_3 depending on the *a priori data* only. The Lipschitz constants of such a transformation and of its inverse are dominated by constants only depending on M, δ and L.

The function $\tilde{f} = f \circ \chi_1^{-1}$ is k_1-quasiconformal, where k_1 depends only on k and on the Lipschitz constants of χ_1 and χ_1^{-1}. Then by a Representation Theorem proved by L. Bers and L. Nirenberg, [11], there exist a k_1-quasiconformal mapping χ_2 from $B_1(0)$ onto itself, with $\chi_2(0) = 0$, and a holomorphic function $\tilde{F} = \tilde{U} + i\tilde{V}$ on $\chi_2(\tilde{D})$ such that the representation $\tilde{f} = \tilde{F} \circ \chi_2$ holds.

By [24, Chapter 3, Theorem 5.2], we may find a conformal mapping χ_3 from $\chi_2(\tilde{D})$ onto a circular domain D still satisfying $\chi_3(0) = 0$ and $\partial B_1(0) = \chi_3(\partial B_1(0))$, $\partial B_1(0)$ being the exterior boundary of D. Then picking $\chi = \chi_3 \circ \chi_2 \circ \chi_1$ and $F = U + iV = \tilde{F} \circ \chi_3^{-1}$ the conclusion is immediate once the following Lemma is available. ∎

LEMMA 3.2. *Let D_0 be a circular domain such that $0 \in D_0$, its exterior boundary is $\partial B_1(0)$ and the minimal radius and separation distance of its multiple cavity are greater than a positive constant d_0. Fixed k, $0 \leqslant k < 1$, there exist constants $d_1 > 0$, $C_2 > 0$ and α_2, $0 < \alpha_2 < 1$, depending on d_0 and k only such that if χ is a k-quasiconformal mapping from D_0 onto another circular domain D_1 whose exterior boundary is $\partial B_1(0)$ such that $\chi(0) = 0$ and $\partial B_1(0) = \chi(\partial B_1(0))$, then the minimal radius and separation distance of the multiple cavity of D_1 are greater than d_1 and χ verifies*

$$|\chi(x) - \chi(y)| \leqslant C_2\, |x - y|^{\alpha_2} \qquad \text{for any} \quad x, y \in D_0 \tag{3.8}$$

and

$$|\chi^{-1}(x) - \chi^{-1}(y)| \leqslant C_2\, |x - y|^{\alpha_2} \qquad \text{for any} \quad x, y \in D_1. \tag{3.9}$$

We defer the rather technical proof of this Lemma to the Appendix.

Let D, F and χ be as in the thesis of Proposition 3.1. Then, by the regularity properties of D and χ, by (2.5)(a) and (2.7) and standard regularity theory we immediately infer

$$|F(z_1)-F(z_2)| \leqslant C_3 \, |z_1-z_2|^{\alpha_3} \qquad \text{for every} \quad z_1, z_2 \in \bar{D}, \tag{3.10}$$

and consequently

$$|f(z_1)-f(z_2)| \leqslant C_4 \, |z_1-z_2|^{\alpha_4} \qquad \text{for every} \quad z_1, z_2 \in \overline{\Omega\setminus\Sigma}, \tag{3.11}$$

where C_3, C_4 and α_3, α_4, $0<\alpha_3, \alpha_4<1$, depend on the *a priori data* only.

We remark that if as usual we extend v on Ω in such a way that $v|_{\sigma_i} = v|_{\partial\sigma_i} = c_i$ for any $i=1, \ldots, n$, then it is easy to show that we have

$$|v(z_1)-v(z_2)| \leqslant C_5 \, |z_1-z_2|^{\alpha_4} \qquad \text{for every} \quad z_1, z_2 \in \bar{\Omega}, \tag{3.12}$$

C_5 depending on the *a priori data* only.

Before stating the following Theorem let us recall that the *Kelvin transform* with respect to the ball $B=B_{r_0}(z_0)$ is given by

$$T_B(z) = \overline{r_0^2/(z-z_0)}+z_0, \qquad z \in \mathbb{C}.$$

THEOREM 3.3. *Under the assumptions of Part* (I) *of Theorem* 2.1, *there exists a positive constant* d_0, *depending on the* a priori data *only, such that for every* $z^0 \in \overline{\Omega\setminus\Sigma}$ *and for every* $d \leqslant d_0$ *there exist finitely many points* $z_k \in \Omega$ *such that for every* $z \in \Omega\setminus\mathring{\Sigma}$ *satisfying* $\operatorname{dist}(z, \partial\Omega) \geqslant d$ *we have*

$$|f(z)-f(z^0)| \geqslant c(d) \prod_k \left(\frac{|z-z_k|}{C_6} \right)^{b_k/\alpha_1}, \tag{3.13}$$

where b_k *are positive integers satisfying*

$$\sum_k b_k \leqslant C(d), \tag{3.14}$$

C_6 *depending on the* a priori data *only,* α_1 *as in* (3.7) *and* $c(d)>0$ *and* $C(d)$ *depending on the* a priori data *and on* d *only.*

Proof. We recall the bi-Lipschitz mapping $\chi_1 \colon \mathbb{C} \mapsto \mathbb{C}$ we considered at the beginning of the proof of Proposition 3.1 which verifies $\chi_1(\Omega\setminus\Sigma)=\tilde{D}$, where $\tilde{D}$ is a circular domain. We have that $\tilde{D}=B_1(0)\setminus\bigcup_{i=1}^n B_{r_i}[x_i]$, $0\in\tilde{D}$, and there exists $\delta_3>0$ depending on the *a priori data* only such that for any $i=1, \ldots, n$ $r_i \geqslant \delta_3$ and $B_{r_i+\delta_3}(x_i)\setminus B_{r_i}[x_i]$ is contained in $\tilde{D}$. The function $\tilde{f}=f\circ\chi_1^{-1}$ which is k_1-quasiconformal, k_1 depending on the *a priori data* only, may be extended to another k_1-quasiconformal function, still denoted by $\tilde{f}$, on the circular domain $\tilde{D}_1 = B_1(0) \setminus \bigcup_{i=1}^{n} B_{lr_i}[x_i]$, where l, $0<l<1$, depends on δ_3 only, in the following way

(3.15)
$$\tilde{f}(z) = \overline{\tilde{f}(T_{B_{r_i}(x_i)}(z))} + 2c_i\mathrm{i} \qquad \text{for any} \quad z \in B_{r_i}(x_i) \setminus B_{lr_i}[x_i],\ i = 1, \ldots, n,$$

where $c_i = v|_{\partial\sigma_i} = \tilde{v}|_{\partial B_{r_i}(z_i)}$.

As in the proof of Proposition 3.1 we apply the Representation Theorem, [11], and Lemma 3.2 to obtain a circular domain D, F, a holomorphic function on D, and a quasiconformal mapping $\chi_2 \colon \tilde{D}_1 \mapsto D$ such that $\tilde{f} = F \circ \chi_2$. We recall that we may assume $D = B_1(0) \setminus \bigcup_{i=1}^{n} B_{s_i}[y_i]$ and that for any $i = 1, \ldots, n$ $s_i \geqslant \delta_2 > 0$, δ_2 depending on the *a priori data* only. Moreover, for any $i = 1, \ldots, n$, we have that $B_{s_i+\delta_2}(y_i) \setminus B_{s_i}[y_i]$ is contained in D. We denote $\chi = \chi_2 \circ \chi_1 : \chi_1^{-1}(\tilde{D}) \mapsto D$ and we remark that χ verifies (3.6), (3.7) on $\chi_1^{-1}(\tilde{D})$ and D respectively and on $\Omega \setminus \Sigma$ we have $f = F \circ \chi$.

It is easy to see that we also have

$$|F(z_1) - F(z_2)| \leqslant C_7\, |z_1 - z_2|^{\alpha_3} \qquad \text{for every} \quad z_1, z_2 \in \bar{D}, \tag{3.16}$$

C_7 depending on the *a priori data* only.

We take $z^0 \in \overline{\Omega \setminus \Sigma}$. Letting $w^0 = \chi(z^0)$, we set $F_0 = F(w^0) = f(z^0)$. Let $Z = \{w_k\}$ be the countable set of the zeroes of $F - F_0$ in D. We have that setting $\phi = \log|F - F_0|$

$$\Delta\phi = 0 \qquad \text{in} \quad D \setminus Z,$$

and since ϕ has negatively diverging isolated singularities at each w_k, there exist positive integers b_k such that, in the sense of distributions,

$$\Delta\phi = 2\pi \sum_k b_k \delta(\cdot - w_k) \qquad \text{in } D.$$

Fixed a positive d we denote

$$D_d = \{z \in D \mid \operatorname{dist}(z, \partial D) > d\}.$$

Then, by arguments in [3] based on Harnack's inequality and the comparison principle, there exist positive constants C_8 and C_8' depending on δ_2 only such that

$$\sum_{w_k \in D_{2d}} b_k \leqslant C_8 d^{-C_8'} \left[1 + \log\left(\frac{\max_{D_d} |F - F_0|}{\max_{D_{2d}} |F - F_0|} \right) \right]. \tag{3.17}$$

Moreover, there exist positive constants C_9, C_9' and C_{10} also depending on δ_2 only such that if we set $c_1(d) = C_9 d^{-C_9'}$, which is greater than 1 if d is small enough, we have for any $w \in D_{3d}$

$$(3.18)\quad |F(w)-F_0| \geqslant \mathrm{e}^{-c_1(d)} \left[\frac{(\max_{D_{3d}} |F-F_0|)^{c_1(d)}}{(\max_{D_{2d}} |F-F_0|)^{c_1(d)-1}} \right] \prod_{w_k \in D_2 d} \left(\frac{|w-w_k|}{C_{10}} \right)^{b_k}.$$

By (3.16) we readily observe that

$$(3.19)\qquad \max_D |F-F_0| \leqslant C_{11},$$

where C_{11} depends on the *a priori data* only. Moreover, if we denote $V_0 = V(z^0)$, we have the following estimate

$$\max_D |F-F_0| \geqslant \max_D |V-V_0| \geqslant \tfrac{1}{2} \operatorname{osc}_D |V|.$$

Then we infer that $\operatorname{osc}_D |V| \geqslant \operatorname{osc}_{\partial B_1(0)} |V|$ and also $\operatorname{osc}_{\partial B_1(0)} |V| = \operatorname{osc}_{\partial\Omega} |v| = \operatorname{osc}_{\partial\Omega} |\Psi|$.

Hence, since $\operatorname{osc}_{\partial\Omega}|\Psi| \geqslant \|\Psi\|_{L^2(\partial\Omega)}/|\partial\Omega|$, by (2.5)(b) and the a priori information on the domain Ω, we can find a positive constant C_{12} depending on the *a priori data* only such that

$$\max_D |F-F_0| \geqslant C_{12}.$$

Again by (3.16) we may find $\tilde{d}_0 > 0$ depending on the *a priori data* only such that for any d, $0 < d \leqslant \tilde{d}_0$, we have

$$(3.20)\qquad \max_{D_{3d}} |F-F_0| \geqslant C_{12}/2.$$

Then by the Hölder continuity properties of χ and its inverse, (3.6) and (3.7), we may find a constant d_0 depending on the *a priori data* only such that for any d, $0 < d \leqslant d_0$, there exists $\tilde{d}$, $0 < \tilde{d} \leqslant \tilde{d}_0$, depending on the Hölder constants of χ and χ^{-1} and on d only, such that for every $z \in \Omega \setminus \mathring{\Sigma}$ satisfying $\operatorname{dist}(z, \partial\Omega) \geqslant d$ we have $w = \chi(z) \in D_{3\tilde{d}}$.

Then, since $|f(z)-f(z^0)| = |F(w)-F_0|$, taking $z_k = \chi^{-1}(w_k)$, by (3.18), (3.19), (3.20) and by (3.7) it follows

$$(3.21)\qquad |f(z)-f(z^0)| \geqslant \mathrm{e}^{-c_1(\tilde{d})} \left[\frac{(C_{12}/2)^{c_1(\tilde{d})}}{(C_{11})^{c_1(\tilde{d})-1}} \right] \prod_k \left(\frac{|z-z_k|}{C_{13}} \right)^{b_k/\alpha_1},$$

where C_{13} depends on the *a priori data* only and, by (3.17), we clearly have

$$\sum_k b_k \leqslant C_8 \tilde{d}^{-C'_8} \left[1 + \log \left(\frac{C_{11}}{C_{12}/2} \right) \right]. \tag{3.22}$$

This clearly concludes the proof. ∎

PROPOSITION 3.4. *Let all the hypotheses of Part* (I) *of Theorem* 2.1, *with the exception of* (2.9), *be satisfied. Let v and v' be the stream functions associated to u and u' respectively. If we have*

$$\|v - v'\|_{L^\infty(\Omega)} \leqslant \eta, \tag{3.23}$$

then the two multiple cavities Σ and Σ' satisfy

$$\mathrm{d}_H(\Sigma, \Sigma') \leqslant K_3 \eta^{\beta_2}, \tag{3.24}$$

$K_3 > 0$, β_2, $0 < \beta_2 < 1$, *depending on the* a priori data *only.*

Proof. Let $p = \mathrm{d}_H(\Sigma, \Sigma')$. Let us assume, without losing the generality, that $p = \sup_{z \in \Sigma'} \mathrm{dist}(z, \Sigma)$.

Then there exist positive constants C_{14} and C_{15}, depending on the *a priori data* only, and a point $z^0 \in \Sigma'$ such that $B_{C_{14}p}(z^0) \subset \Sigma'$ and for any $w \in B_{C_{14}p}(z^0)$ we have $\mathrm{dist}(w, \Sigma) \geqslant C_{15}p$. Since $B_{C_{14}p}(z^0) \subset \Sigma'$, recalling (2.3), clearly we also have $\mathrm{dist}(w, \partial\Omega) \geqslant \delta$ for any $w \in B_{C_{14}p}(z^0)$.

By the maximum principle, the level set $\{u = u(z^0)\}$ contains a continuum containing z^0 and intersecting $\partial B_{C_{14}p}(z^0)$ in at least two different points. Let us fix $d = \min\{d_0, \delta\}$, d_0 as in Theorem 3.3. Let us consider the points z_k obtained in Theorem 3.3 with respect to the point z^0 and the positive number d. Their number, by (3.14), is bounded by a constant N depending on the *a priori data* only. There exists a constant $C_{16} > 0$ depending on N and on C_{14} only such that we may find $N+1$ pairwise disjoint open discs with radius $C_{16}p$ that are contained in $B_{C_{14}p}(z^0)$ and whose center belongs to $\{u = u(z^0)\}$. Therefore at least one of these discs has none of the points z_k in its interior. Let z^1 be the center of this disc. Clearly for any z_k we have $|z^1 - z_k| \geqslant C_{16}p$.

Then by (3.13) we have

$$|f(z^1) - f(z^0)| \geqslant c(d) \prod_k \left(\frac{|z^1 - z_k|}{C_6} \right)^{b_k/\alpha_1};$$

hence, by (3.14) and since $|z^1 - z_k| \geqslant C_{16}p$,

$$|f(z^1) - f(z^0)| \geqslant c(d)\left(\frac{C_{16}p}{C_6}\right)^{C(d)/\alpha_1}.$$

Since we have that $u(z^1) = u(z^0)$ and, obviously $v(z^1) = v(z^0)$, we deduce

$$|f(z^0) - f(z^1)| = |v(z^0) - v(z^1)| \leqslant |v(z^0) - v'(z^0)| + |v(z^1) - v'(z^1)| \leqslant 2\eta.$$

Putting together the last two equations the conclusion easily follows. ∎

Let us denote $\Phi = W + \mathrm{i}Z = u - u' + \mathrm{i}(v - v') : \Omega \backslash (\Sigma \cup \Sigma') \mapsto \mathbb{C}$.

We can normalize Z in order to have that it is identically zero on $\partial\Omega$. Moreover by (2.9) we obtain $|W| \leqslant \varepsilon$ on Γ.

Recalling (3.11) there exists a constant D_1 depending on the *a priori data* only such that

$$|\Phi(z)| \leqslant D_1 \qquad \text{for any} \quad z \in \Omega \backslash (\Sigma \cup \Sigma'). \tag{3.25}$$

We shall consider the following Cauchy type problem

$$\begin{cases} \Phi_{\bar{z}} = \mu \Phi_z + \nu \overline{\Phi_z} & \text{in} \quad \Omega \backslash (\Sigma \cup \Sigma'), \\ |\Phi| \leqslant \varepsilon & \text{on } \Gamma, \\ \Im \Phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.26}$$

where $|\mu| + |\nu| \leqslant k < 1$.

Recalling Proposition 3.4, the stability estimate on the inverse problem of cavities has been reduced to a stability estimate for the Cauchy type problem (3.26), that is obtaining an upper bound for $|Z|$ on $\bar{\Omega}$ in terms of the boundary error ε.

We shall obtain different kinds of stability estimates for the Cauchy type problem (3.26), depending on the assumptions stated in the different parts of Theorem 2.1.

PROPOSITION 3.5. *Let the assumptions of Part* (I) *of Theorem* 2.1 *be satisfied and let v and v' be the stream functions associated to u and u' respectively. Then we have*

$$\|v - v'\|_{L^\infty(\bar{\Omega})} \leqslant \eta(\varepsilon), \tag{3.27}$$

where $\eta\colon (0,+\infty)\mapsto(0,+\infty)$ *satisfies*

$$\eta(\varepsilon)\leqslant K_4(\log|\log\varepsilon|)^{-\beta_3}\qquad \textit{for every } \varepsilon,\quad 0<\varepsilon<1/e, \tag{3.28}$$

where K_4 *and* $\beta_3>0$ *depend on the* a priori data *only*.

Proof. We give a sketch of the proof which is based on a technique developed in [4] (see also [8]). The main difference here is the presence of a multiple cavity, instead of a single crack.

First of all we define, as in [4], the following kind of so-called h-tubes. If $z_0\in\Gamma$, let l be the segment bisecting the open angular sector $S\subset\Omega$ whose vertex is z_0, whose radius is δ and whose amplitude depends on M only. We know that $\operatorname{dist}(z_1,\partial\Omega)\geqslant M_2\,|z_0-z_1|$, for any $z_1\in l$, $M_2<1$ depending on M only.

Let γ be a smooth curve contained in $\Omega\backslash(\Sigma\cup\Sigma')$ so that its first end-point z_0 belongs to Γ, γ coincide with l for a length of at least h and thereafter the distance of any point of γ from $\partial\Omega$ is greater than M_2h. An h-tube will be the M_2h neighbourhood of any curve $\tilde\gamma$ obtained by removing from such a curve γ its linear part of length h which is contained in l.

An h-accessible point will be a point belonging to the closure of an h-tube which is contained in $\Omega\backslash(\Sigma\cup\Sigma')$. We denote with G_h the set of h-accessible points.

If we apply the method used in [4] together with Theorem 4.5 in [8], we obtain for every $z\in G_h$ and every h, $0<h\leqslant h_0$,

$$|v(z)-v'(z)|\leqslant D_2h^{\alpha_4}+(D_3+\varepsilon)\left(\frac{\varepsilon}{D_3+\varepsilon}\right)^{\exp(-D_4/h^2)}, \tag{3.29}$$

with constants D_2, D_3, D_4, h_0 depending on the *a priori data* only and α_4 as in (3.12).

Given the Hölder continuity of v and of v', which is stated in (3.12), and the maximum principle, we may extend the estimate (3.29) to any $z\in\bar\Omega$ applying the method described in the proof of Theorem 3.1 in [4] with few modifications. In particular the main difference is that for any connected component Q of $\Omega\backslash G_h$ more than two connected components of Σ or Σ' may be involved. However, in this case, there exists a constant D_5 depending on the *a priori data* only such that for any connected component σ of Σ or Σ' contained in Q with at least one point belonging to ∂G_h, we may find a point $w_0\in\sigma\cap\partial G_h$ and a point $w_1\in\partial G_h$ belonging to another connected component of Σ or Σ' contained in Q, such that $|w_0-w_1|\leqslant D_5h$. For analogous considerations see [4, Lemma 3.6]. Then, by an iterated use of the above inequality and by the maximum principle, we find that there exists a constant c depending on Q such that if $\tilde c$ is the constant value of v

(or respectively v') on any connected component of Σ (respectively Σ') contained in Q then we have

$$|\tilde{c}-c| \leqslant D_6 h^{\alpha_4} + (D_3+\varepsilon)\left(\frac{\varepsilon}{D_3+\varepsilon}\right)^{\exp(-D_4/h^2)}, \qquad \text{for every } h, \quad 0<h\leqslant h_0,$$

D_6 depending on the *a priori data* only. Obtained this result we conclude as in [4].

Once (3.29) is available for any $z \in \bar{\Omega}$ the thesis easily follows. ▮

PROPOSITION 3.6. *Let the hypothesis of Part* (II) *of Theorem* 2.1 *be satisfied. Then v and v', the stream functions associated to u and u' respectively, verify* (3.27) *where* $\eta\colon (0,+\infty) \mapsto (0,+\infty)$ *satisfies*

$$\eta(\varepsilon) \leqslant K_5\,|\log \varepsilon|^{-\beta_4} \qquad \textit{for every } \varepsilon, \quad 0<\varepsilon<1/\mathrm{e}, \tag{3.30}$$

K_5 *and* $\beta_4>0$ *depending on the* a priori data *only.*

Proof. Let G be the connected component of $\Omega\backslash(\Sigma\cup\Sigma')$ such that $\Gamma\subset\partial G$. Since γ_i and γ'_j are RLG then it is not difficult to show that G satisfies a uniform interior cone condition, that is for any point $z\in\partial G$ there exists an angular sector S contained in G, with vertex in z and whose positive radius and amplitude depend on the *a priori data* only and do not depend on z.

Therefore by the technique developed in [21] we are able to obtain

$$|v(z)-v'(z)| \leqslant D_7\,|\log \varepsilon|^{-\alpha_5} \qquad \text{for every} \quad z\in G, \tag{3.31}$$

where D_7 and $\alpha_5>0$ depend on the *a priori data* only. Then, again with the help of the maximum principle, the conclusion follows. ▮

LEMMA 3.7. *Let us fix a positive integer k and a constant α, $0<\alpha\leqslant 1$, and let $\Gamma_0=\bigcup_{i=1}^n \gamma_i$ and $\Gamma'_0=\bigcup_{j=1}^m \gamma'_j$ be two finite families of simple closed curves, such that the domains bounded by each of the curves of one of the two families are pairwise disjoint. We assume that the two families are both $C^{k,\alpha}$ with constants δ, M, and the length of any curve belonging to one of the two families is bounded by a constant L.*

Then there exists $p_0>0$ depending on δ, M, L, k and α only such that if $p=d_H(\Gamma_0,\Gamma'_0)\leqslant p_0$ then both Γ_0 and Γ'_0 have n connected components, which ordered in a suitable way verify

$$d_H(\gamma_i,\gamma'_i) \leqslant p \qquad \textit{for any} \quad i=1,\dots,n. \tag{3.32}$$

Furthermore for any $i=1, \ldots, n$, there exist regular parametrisations $z_i = z_i(t)$ and $z_i' = z_i'(t)$, $0 \leqslant t \leqslant 1$, of γ_i and γ_i' respectively such that for every $\tilde{\alpha}$, $0 < \tilde{\alpha} < \alpha$,

$$\|z_i - z_i'\|_{C^{k,\tilde{\alpha}}[0,1]} \leqslant K_6(d_H(\gamma_i, \gamma_i'))^{(\alpha-\tilde{\alpha})/(k+\alpha)}, \tag{3.33}$$

where K_6 depends on δ, M, L, k, α and on $\tilde{\alpha}$ only.

Proof. By our assumptions we have that both the families verify

$$\operatorname{dist}(\gamma_i, \gamma_j) \geqslant \delta_4, \qquad \text{for every} \quad i \neq j,$$

with a constant $\delta_4 > 0$ depending on δ, M, L, k and α only.

Therefore the first part of the Lemma is obvious. Once the first part is established, the second one may be obtained following a procedure analogous to the one used to prove Lemma 2.1 in [21]. ∎

Proof of Theorem 2.1. Concerning Part (I) of Theorem 2.1, (2.10) and (2.11) are a direct consequence of Proposition 3.4 and of Proposition 3.5, whereas (2.12) may be deduce from (2.10) and (2.11) by taking into account (3.1).

The Part (II) may be obtained through Proposition 3.4 and Proposition 3.6.

For what concerns Part (III) the proof is an easy consequence of the previous part of Theorem 2.1 and of Lemma 3.7. In fact we have that the two families of curves $\Gamma_0 = \bigcup_{i=1}^{n} \gamma_i$ and $\Gamma_0' = \bigcup_{j=1}^{m} \gamma_j'$ which consist of the boundaries of the connected components of Σ and Σ' respectively satisfy the assumptions of Lemma 3.7.

Then if ε is small enough we have, by Part (I) of Theorem 2.1, $d_H(\Gamma_0, \Gamma_0') \leqslant p_0$ and hence the number of connected components of Σ and Σ' is the same.

Given (3.32), (3.33), it is not difficult to show that there exists $\varepsilon_1 > 0$, depending on the *a priori data* and on k and α only such that if $\varepsilon \leqslant \varepsilon_1$ then γ_i, $i=1, \ldots, n$, and γ_j', $j=1, \ldots, m$, are RLG with constants $\delta_5 > 0$, $M_3 > 0$ with δ_5 and M_3 depending on δ, M, L, k and α only and not depending on ε.

So the conclusion follows. ∎

4. INSTABILITY EXAMPLE

Let $\Omega = B_1(0)$ and let $\sigma_0 = B_{1/2}[0]$. Let $D_0 = \Omega \backslash \sigma_0$. The two connected components of the boundary of D_0 are the two simple closed curves $\beta = \partial\Omega = \partial B_1(0)$ and $\gamma_0 = \partial\sigma_0 = \partial B_{1/2}(0)$.

For any $n=1, 2, \ldots,$ let us denote by f_n the holomorphic function so defined

$$f_n(z) = z \exp[\epsilon_n(z^n - z^{-n})], \qquad z \in \mathbb{C}\setminus\{0\}, n=1, 2, \ldots, \tag{4.1}$$

where ϵ_n is the following positive real constant

$$\epsilon_n = \frac{C_0}{n^k 2^n}, \tag{4.2}$$

where k is a fixed positive integer and C_0 is a positive constant to be chosen later.

The first derivative of f_n is given by

$$f'_n(z) = [1 + \epsilon_n n(z^n - z^{-n})] \exp[\epsilon_n(z^n - z^{-n})], \qquad z \in \mathbb{C}\setminus\{0\}, n=1, 2, \ldots,$$

hence we may find a positive constant C_0, C_0 not depending on n and on k, such that if (4.2) holds then we have

$$|f'_n(z) - 1| \leqslant 1/4, \qquad \text{for any} \quad z \in \overline{D_0}, n=1, 2, \ldots, \tag{4.3}$$

and therefore f_n is invertible on a neighbourhood (which may depend on n) of $\overline{D_0}$.

From now on we shall assume that this condition is satisfied. For any $n=1, 2, \ldots,$ we call $D_n = f_n(D_0)$. The boundary of D_n has two connected components, the image through f_n of β and γ_0 respectively. It is easily seen that $f_n(\beta) = \beta$ and we shall denote by γ_n the image through f_n of γ_0. We remark that γ_n is a Jordan curve and we denote by σ_n the closed region bounded by γ_n. Therefore we have that $D_n = \Omega\setminus\sigma_n$.

By switching to polar coordinates, we shall characterize more precisely the behaviour of f_n along β and γ_0 and hence the regularity properties of γ_n and, consequently, of σ_n.

Let us introduce polar coordinates in the following way. Given $z \in \mathbb{C}\setminus\{0\}$ let (ρ, θ), $\rho > 0$, satisfy $z = \rho \exp(i\theta)$. We have that $\rho = |z|$ and θ is defined up to equivalence modulus 2π. We call (ρ, θ) the polar coordinates of z. Then, in these coordinates, f_n can be written as

$$f_n(\rho, \theta) = (\varphi_n(\rho, \theta), \phi_n(\rho, \theta)),$$

where

$$\varphi_n(\rho, \theta) = \rho \exp[\epsilon_n(\rho^n - \rho^{-n}) \cos n\theta]$$

and

$$\phi_n(\rho, \theta) = \theta + \epsilon_n(\rho^n + \rho^{-n}) \sin n\theta.$$

First of all we notice that if $\rho = 1$ then $\varphi_n(1, \theta) = 1$ for any $\theta \in \mathbb{R}$ and we have

$$|\phi_n(1, \theta) - \theta| \leqslant 2\epsilon_n, \qquad \text{for any} \quad \theta \in \mathbb{R}. \tag{4.4}$$

Then we want to estimate the Hausdorff distance between σ_n and σ_0. It is easy to observe that

$$\mathrm{d}_H(\sigma_n, \sigma_0) = \max_{[0, 2\pi]} |\varphi_n(1/2, \theta) - 1/2|.$$

We may find two constants C_1 and C_2, $0 < C_1 < C_2$, such that

$$0 < C_1 \epsilon_n 2^n \leqslant \mathrm{d}_H(\sigma_n, \sigma_0) \leqslant C_2 \epsilon_n 2^n. \tag{4.5}$$

Without loss of generality, changing C_0 in (4.2) if necessary, we may assume $C_2 \epsilon_n 2^n \leqslant 1/4$.

Let us fix ρ, $1/2 \leqslant \rho \leqslant 1$, and let us consider the function $\phi_n(\rho, \cdot)\colon [0, 2\pi] \mapsto \mathbb{R}$. Then we can find $C_0 > 0$ not depending on n and on k such that if (4.2) holds then $|\frac{\partial}{\partial\theta} \phi_n(\rho, \theta) - 1| \leqslant 1/3$ for any $\rho \in [1/2, 1]$ and $\theta \in [0, 2\pi]$. By this estimate we infer that $\phi_n(\rho, \cdot)\colon [0, 2\pi] \mapsto [0, 2\pi]$ is bi-Lipschitzian with Lipschitz constants not depending on n, on k and on ρ.

Moreover, for any integer $i \geqslant 2$ we notice that

$$\left| \frac{\partial^i}{\partial \theta^i} \phi_n(\rho, \theta) \right| \leqslant \epsilon_n(\rho^n + \rho^{-n})\, n^i.$$

If we fix the positive integer k and we define ϵ_n as in (4.2) with $C_0 > 0$ satisfying the previously stated conditions, it is straightforward to prove that for any $n = 1, 2, \ldots,$ γ_n is a C^k simple closed curve with constants δ, M not depending on n. Here the notion of a C^k curve with constants δ, M is in the sense specified at the beginning of Section 2, with the obvious modification of replacing the $C^{k,\alpha}$ norm with the one in C^k.

For any $n = 0, 1, 2, \ldots,$ let us consider, as usual, the following Sobolev spaces $H^1(D_n) = \{u \in L^2(D_n) \mid \nabla u \in L^2(D_n)\}$. We denote by $H^{1/2}(\beta)$ its corresponding trace space on β. By $H^{-1/2}(\beta)$ we shall denote the dual space to $H^{1/2}(\beta)$. With ${}_0H^{1/2}(\beta)$ and ${}_0H^{-1/2}(\beta)$ the corresponding subspaces of elements with zero means are considered. We remark that ${}_0H^{1/2}(\beta)$ and ${}_0H^{-1/2}(\beta)$ are dual to each other. With ${}_0L^2(\beta)$ we denote the L^2 functions on β with zero average. We remark that the dual of ${}_0L^2(\beta)$ is the space itself. Finally, if X and Y are two Banach spaces we shall denote by $B(X, Y)$ the space of all bounded linear operators from X to Y, with the usual norm.

Concerning trace spaces, fractional Sobolev spaces and interpolation inequalities, which will be used several times in the sequel, we refer to [1] and [19].

Let $\eta \in {}_0H^{-1/2}(\beta)$. Then for any $n = 0, 1, 2, \ldots,$ let us consider the following Neumann type boundary value problem

$$
(\mathrm{NP}_n)\qquad \begin{aligned}
&\Delta u_n = 0 && \text{in } D_n,\\
&\nabla u_n \cdot \nu = 0 && \text{on } \gamma_n,\\
&\nabla u_n \cdot \nu = \eta && \text{on } \beta,\\
&u_n|_\beta \in {}_0H^{1/2}(\beta).
\end{aligned}
$$

The weak formulation of the problem is the following. To find $u_n \in H^1(D_n)$ such that $u_n|_\beta \in {}_0H^{1/2}(\beta)$ and the following holds

$$
(\mathrm{NP}'_n)\qquad \int_{D_n} \nabla u_n \nabla \phi = \eta[\phi|_\beta], \qquad \text{for any} \quad \phi \in H^1(D_n).
$$

We have that the solution to (NP_n) exists and is unique and we may find a constant C not depending on n such that if $D = B_1 \setminus \overline{B_{4/5}}$ then

$$
\|u_n\|_{H^1(D)} \leqslant C\|\eta\|_{H^{-1/2}(\beta)}. \tag{4.6}
$$

For any $n = 0, 1, 2, \ldots,$ let $N_n\colon {}_0H^{-1/2}(\beta) \mapsto {}_0H^{1/2}(\beta)$ be the Neumann-to-Dirichlet map defined in the following way

$$
N_n(\eta) = u_n|_\beta \qquad \text{for any} \quad \eta \in {}_0H^{-1/2}(\beta), \tag{4.7}
$$

where u_n is the solution to (NP_n).

From (4.6) we have that

$$
\|N_n(\eta)\|_{{}_0H^{1/2}(\beta)} \leqslant C\,\|\eta\|_{{}_0H^{-1/2}(\beta)}, \qquad \text{for any} \quad \eta \in {}_0H^{-1/2}(\beta), \tag{4.8}
$$

where C is a positive constant which does not depend on n.

Let us state our instability result.

THEOREM 4.1. *Let us fix a positive integer k. Then there exists a constant $C_0 > 0$ such that if* (4.2) *holds then for any $n = 0, 1, 2, \ldots,$ γ_n is a C^k simple* *closed curve with positive constants δ, M not depending on n and the following inequality holds*

$$d_H(\sigma_n, \sigma_0) \geqslant C\,|\log \|N_n - N_0\|_{B({}_0H^{-1/2}(\beta),\, {}_0H^{1/2}(\beta))}|^{-k}, \tag{4.9}$$

where C is a positive constant which does not depend on n.

Remark. Let us observe that in inequality (4.9) some kind of dependence on k, the number of derivatives of the curves γ_n which are *a priori* uniformly bounded, should be expected. In fact, in a similar setting, [13], Hölder type dependence on a suitably chosen boundary measurement was proved if an analyticity condition on the unknown curve γ holds.

The proof of Theorem 4.1 will be obtained through three lemmas.

LEMMA 4.2. *There exists a positive constant C such that for any $\eta \in {}_0L^2(\beta)$ we have*

$$\|N_0(\eta)\|_{H^1(\beta)} \leqslant C\,\|\eta\|_{L^2(\beta)}. \tag{4.10}$$

Proof. We have already observed, (4.8), that

$$\|N_0(\eta)\|_{{}_0H^{1/2}(\beta)} \leqslant C\,\|\eta\|_{{}_0H^{-1/2}(\beta)}, \qquad \text{for any} \quad \eta \in {}_0H^{-1/2}(\beta). \tag{4.11}$$

Moreover it is not difficult to show that if u_0 is the solution to (NP_0) then we have, for a positive constant C,

$$\|u_0\|_{H^1(D_0)} \leqslant C\,\|\eta\|_{{}_0H^{-1/2}(\beta)}, \qquad \text{for any} \quad \eta \in {}_0H^{-1/2}(\beta).$$

By standard regularity results, see for instance [23], we have that if $\eta \in {}_0H^{1/2}(\beta)$ then u_0 belongs to $H^2(D_0)$ and the following estimate holds

$$\|u_0\|_{H^2(D_0)} \leqslant C(\|\eta\|_{{}_0H^{1/2}(\beta)} + \|u_0\|_{H^1(D_0)}), \qquad \text{for any} \quad \eta \in {}_0H^{1/2}(\beta).$$

Then we immediately deduce

$$\|N_0(\eta)\|_{H^{3/2}(\beta)} \leqslant C\,\|\eta\|_{{}_0H^{1/2}(\beta)}, \qquad \text{for any} \quad \eta \in {}_0H^{1/2}(\beta). \tag{4.12}$$

Therefore the thesis may be obtained through (4.11) and (4.12) by using standard interpolation inequalities. ∎

LEMMA 4.3. *There exists a positive constant C not depending on n such that*

(4.13)

$$\|(N_n - N_0)(\eta)\|_{L^2(\beta)} \leqslant C\epsilon_n^{1/2}\,\|\eta\|_{L^2(\beta)}, \qquad \textit{for any} \quad \eta \in {}_0L^2(\beta),\, n = 1, 2, \ldots.$$

Proof. For any $n = 0, 1, 2, \ldots,$ let us consider the linear operator $N_n: {}_0L^2(\beta) \mapsto {}_0L^2(\beta)$. We have that N_n, with respect to these two spaces, is bounded and self-adjoint. This can be easily deduced by the weak formulation of our boundary value problem, (NP'_n).

Let $h_n: D_n \mapsto D_0$ be the inverse map of f_n, then h_n can be extended to the closure of D_n and let us recall some properties of the restriction of h_n to β.

We have that $h_n|_\beta: \beta \mapsto \beta$ is invertible, bi-Lipschitz with constants not depending on n and the following estimates holds

$$|h_n(z) - z| \leqslant C\epsilon_n, \tag{4.14}$$

where C does not depend on n.

For any $n = 1, 2, \ldots,$ let us define the linear operator $T_n: L^2(\beta) \mapsto L^2(\beta)$ in the following way

$$T_n(\eta)(z) = \eta(h_n(z)), \qquad \text{for any} \quad z \in \beta, \eta \in L^2(\beta).$$

These linear operators are continuous with norm independent on n, that is

$$\|T_n(\eta)\|_{L^2(\beta)} \leqslant C\, \|\eta\|_{L^2(\beta)}, \qquad \text{for any} \quad \eta \in L^2(\beta), n = 1, 2, \ldots, \tag{4.15}$$

they are invertible,

$$(T_n)^{-1}(\eta)(z) = \eta(f_n(z)), \qquad \text{for any} \quad z \in \beta, \eta \in L^2(\beta),$$

and their inverses are continuous with norm independent on n.

Let T_n^* be the adjoint operator to T_n, $n = 1, 2, \ldots,$ then $T_n^*: L^2(\beta) \mapsto L^2(\beta)$ is defined

$$T_n^*(\eta) = (T_n)^{-1}\left(\eta \frac{1}{|h'_n|}\right), \qquad \text{for any} \quad \eta \in L^2(\beta).$$

Finally let us observe that if $\eta \in {}_0L^2(\beta)$ then also $T_n^*(\eta) \in {}_0L^2(\beta)$.

Let $P: L^2(\beta) \mapsto {}_0L^2(\beta)$ be the projection of $L^2(\beta)$ onto ${}_0L^2(\beta)$ given by

$$P(\eta) = \eta - \frac{1}{2\pi}\int_\beta \eta, \qquad \text{for any} \quad \eta \in L^2(\beta).$$

Clearly P is a linear bounded operator with norm 1.

We claim that the following representation holds

$$N_n(\eta) = P[T_n N_0 T_n^*](\eta), \qquad \text{for any} \quad \eta \in {}_0L^2(\beta). \tag{4.16}$$

Let u_n be the solution to (NP_n) with Neumann datum $\eta \in {}_0L^2(\beta)$. Let us denote $v_n = u_n \circ f_n$. Then v_n solves

$$\begin{aligned} \Delta v_n &= 0 && \text{in } D_0, \\ \nabla v_n \cdot \nu &= 0 && \text{on } \gamma_0, \\ \nabla v_n \cdot \nu &= T_n^* \eta && \text{on } \beta. \end{aligned} \tag{4.17}$$

Therefore we have $u_n|_\beta = T_n(v_n|_\beta)$ and $v_n|_\beta$ is equal to $N_0 T_n^*(\eta)$ up to an additive constant. Hence $N_n(\eta) = u_n|_\beta = T_n N_0 T_n^*(\eta) + c_n$.

By the fact that $N_n(\eta) \in {}_0L^2(\beta)$ we can immediately infer that $c_n = -\frac{1}{2\pi} \int_\beta T_n N_0 T_n^*(\eta)$ and hence (4.16) follows.

Now let us take $\psi \in H^1(\beta)$. We want to estimate $\|(T_n - I)(\psi)\|_{L^2(\beta)}$. We have that

$$\|(T_n - I)(\psi)\|^2_{L^2(\beta)} = \int_0^{2\pi} |\psi(h_n(\theta)) - \psi(\theta)|^2 \, d\theta.$$

Then by (4.14) we deduce that $|\psi(h_n(\theta)) - \psi(\theta)| \leqslant C\epsilon_n^{1/2} \|\psi\|_{H^1(\beta)}$ and hence we obtain

$$\|(T_n - I)(\psi)\|_{L^2(\beta)} \leqslant C\epsilon_n^{1/2} \|\psi\|_{H^1(\beta)}, \qquad \text{for any} \quad \eta \in H^1(\beta), \tag{4.18}$$

C not depending on n.

Therefore by Lemma 4.2 we may find a constant C which does not depend on n such that

$$\|(T_n N_0 - N_0)(\eta)\|_{L^2(\beta)} \leqslant C\epsilon_n^{1/2} \|\eta\|_{L^2(\beta)}, \qquad \text{for any} \quad \eta \in {}_0L^2(\beta). \tag{4.19}$$

By duality we have, with the same constant C,

$$\|(N_0 T_n^* - N_0)(\eta)\|_{L^2(\beta)} \leqslant C\epsilon_n^{1/2} \|\eta\|_{L^2(\beta)}, \qquad \text{for any} \quad \eta \in {}_0L^2(\beta). \tag{4.20}$$

Obviously $PN_0 = N_0$, then $N_n - N_0 = P(T_n N_0 T_n^* - N_0)$ and hence for any $\eta \in {}_0L^2(\beta)$ we have

$$\|(N_n - N_0)(\eta)\|_{L^2(\beta)} \leqslant \|(T_n N_0 T_n^* - N_0)(\eta)\|_{L^2(\beta)}.$$

Since

$$\|(T_n N_0 T_n^* - N_0)(\eta)\|_{L^2(\beta)} \leqslant \|T_n(N_0 T_n^* - N_0)(\eta)\|_{L^2(\beta)} + \|(T_n N_0 - N_0)(\eta)\|_{L^2(\beta)}$$

the thesis follows from (4.15), (4.19) and (4.20). ∎

LEMMA 4.4. *$N_n - N_0$ is an infinitely smoothing operator, that is for any positive integer i there exists a constant $C = C(i)$ not depending on n such that we have*

$$\|(N_n - N_0)(\eta)\|_{H^i(\beta)} \leqslant C(i)\, \|\eta\|_{{}_0H^{-1/2}(\beta)}, \qquad \textit{for any} \quad \eta \in {}_0H^{-1/2}(\beta).$$

Proof. Let us fix $\eta \in {}_0H^{-1/2}(\beta)$ and let u_n and u_0 be the solutions to (NP_n) and (NP_0) respectively. By (4.6) and the mean value property of harmonic functions it is clear that for any z such that $|z| = 7/8$ there exists a constant C not depending on n and on η such that

$$|(u_n - u_0)(z)| \leqslant C\, \|\eta\|_{{}_0H^{-1/2}(\beta)}. \tag{4.21}$$

Then we notice that along β $u_n - u_0$ satisfies a homogeneous Neumann condition. Therefore we may extend $u_n - u_0$ on $B_{8/7} \setminus \overline{B_{7/8}}$ according to the following reflection rule

$$(u_n - u_0)(z) = (u_n - u_0)(1/\bar{z}), \qquad \text{for any} \quad z \in B_{8/7} \setminus \overline{B_{7/8}}.$$

We have that $u_n - u_0$ is harmonic in $B_{8/7} \setminus \overline{B_{7/8}}$, by the maximum principle and (4.21), on the same domain is bounded by $C\, \|\eta\|_{{}_0H^{-1/2}(\beta)}$, therefore the thesis easily follows. ∎

Proof of Theorem 4.1. By Lemma 4.3 and Lemma 4.4 applied with $i = 2$ and standard interpolation results we immediately infer

$$\|(N_n - N_0)(\eta)\|_{H^1(\beta)} \leqslant C\epsilon_n^{1/4}\, \|\eta\|_{L^2(\beta)}, \qquad \text{for any} \quad \eta \in {}_0L^2(\beta).$$

By duality we have

$$\|(N_n - N_0)(\eta)\|_{L^2(\beta)} \leqslant C\epsilon_n^{1/4}\, \|\eta\|_{H^{-1}(\beta)}, \qquad \text{for any} \quad \eta \in {}_0H^{-1}(\beta).$$

Then, again by interpolation inequalities, we deduce

$$\|(N_n - N_0)(\eta)\|_{{}_0H^{1/2}(\beta)} \leqslant C\epsilon_n^{1/4}\, \|\eta\|_{{}_0H^{-1/2}(\beta)}, \qquad \text{for any} \quad \eta \in {}_0H^{-1/2}(\beta),$$

with C a constant not depending on n.

Then the thesis may be obtained through a straightforward computation by recalling the definition of ϵ_n, (4.2), and the lower bound on $\mathrm{d}_H(\sigma_n, \sigma_0)$, (4.5). ∎

APPENDIX

Proof of Lemma 3.2. During the proof of this Lemma we shall make use of the notion of capacity. Concerning its definition and its basic properties we refer to [16]. Here let us simply state some notations and the definition. Given a bounded domain D and E a subset of D, the pair (E, D) will be called a *condenser* and we denote by $\operatorname{cap}(E, D)$ the *capacity* of the condenser (E, D). If E is compact then

$$\operatorname{cap}(E, D) = \inf_{u \in W(E, D)} \int_D |\nabla u|^2,$$

where

$$W(E, D) = \{u \in C_0^\infty(D) \mid u \geqslant 1 \text{ on } E\}.$$

Then for any subset E the capacity is defined as

$$\operatorname{cap}(E, D) = \inf_{\substack{E \subset G \subset D \\ G \text{ open}}} \sup_{\substack{K \subset G \\ K \text{ compact}}} \operatorname{cap}(K, D).$$

We note also that the capacity may be computed explicitly if the condenser is an annulus. In fact, see again [16, page 35], we have for $0 < r < R$

$$\operatorname{cap}(B_r[x], B_R(x)) = 2\pi \left(\log \frac{R}{r}\right)^{-1}. \tag{A1}$$

Let $D_0 = B_1(0) \setminus \bigcup_{i=1}^n B_{r_i}[z_i]$ and $D_1 = B_1(0) \setminus \bigcup_{i=1}^n B_{s_i}[w_i]$. We recall that $\chi(\partial B_1(0)) = \partial B_1(0)$ and we have ordered the cavities in such a way that $\chi(\partial B_{r_i}(z_i)) = \partial B_{s_i}(w_i)$ for any $i = 1, \ldots, n$. We note also that, since the minimal radius is bounded from below by $d_0 > 0$, if n denotes the number of connected components of the multiple cavity of D_0 (and obviously also of the one of D_1), we have

$$n \leqslant N, \tag{A2}$$

N depending only on d_0.

We denote $I = \{1, \ldots, n\}$. Then, by the lower bound on the minimal radius and on the separation distance of the multiple cavity of D_0, by (A1) and by elementary properties of capacity, we may find two constants $0 < C_1 < C_2$ depending on d_0 only such that for every I_1, nonempty subset of I, we have

$$0 < C_1 \leqslant \operatorname{cap}\left(\bigcup_{i \in I_1} B_{r_i}[z_i], B_1(0) \Big\backslash \bigcup_{j \in I \setminus I_1} B_{r_j}[z_j] \right) \leqslant C_2. \tag{A3}$$

Since χ is k-quasiconformal then there exists a constant $C_3>0$ depending on k only such that

$$\text{(A4)}\quad 0<C_1/C_3\leqslant \operatorname{cap}\left(\bigcup_{i\in I_1} B_{s_i}[w_i],\, B_1(0)\Big\backslash \bigcup_{j\in I\backslash I_1} B_{s_j}[w_j]\right)\leqslant C_3C_2.$$

holds for any $I_1\subset I$, $I_1\neq\varnothing$, see [16, page 288].

We now claim the following result.

Claim. Given k, $0\leqslant k<1$, and $\delta_0>0$, let f be a k-quasiconformal mapping from the annulus $B_1(0)\backslash B_{1-\delta_0}[0]$ onto $B_1(0)\backslash\sigma$, σ being a closed subset of $B_1(0)$, satisfying $f(\partial B_1(0))=\partial B_1(0)$ and $0\in\sigma$. Then

$$\text{(A5)}\qquad \operatorname{dist}(\sigma,\partial B_1(0))\geqslant\delta_1$$

where $\delta_1>0$ depends on k and δ_0 only.

By the Representation Theorem in [11, page 116] it is enough to prove the Claim when $f=u+iv$ is conformal. Since $0\in\sigma$, by (A1) and the invariance of capacity through conformal mapping, we may find $\delta_2>0$ small enough such that for any $0<\delta_0\leqslant\delta_2$ either the oscillation of u or of v on $\partial B_{1-\delta_0}(0)$ is greater than $1/4$. Then by [2, Theorem 1.3] (see also [8, page 336]) there exist a constant $\delta_3>0$ and a constant $C>0$, both depending on k and δ_0 only, such that if $0<\delta_0\leqslant\delta_3$ we have

$$\text{(A6)}\qquad |f'(z)|\geqslant C,\quad \text{for any } z\in B_{1-\delta_0/4}[0]\backslash B_{1-3\delta_0/4}(0),$$

and from this the conclusion of the proof of the Claim follows very easily.

By the Claim we may immediately infer that there exists a constant d_2 depending on k and d_0 only such that we have

$$\text{(A7)}\qquad \operatorname{dist}(B_{s_i}[w_i],\partial B_1(0))\geqslant d_2\qquad \text{for any}\quad i=1,\dots,n.$$

Let us denote as before

$$\delta_i=\operatorname{dist}\left(B_{s_i}[w_i],\bigcup_{j\neq i} B_{s_j}[w_j]\cup\partial B_1(0)\right),\qquad i=1,\dots,n.$$

Then, for any $i=1,\dots,n$, we consider the following change of coordinates

$$T_i(z)=r_i/(\overline{z-z_i}),\qquad S_i(z)=s_i/(\overline{z-w_i})$$

and we take the function $f_i\colon T_i(D_0)\mapsto S_i(D_1)$ given by

$$f_i=S_i\circ\chi\circ T_i^{-1}.$$

We have that there exists a $\delta_0 > 0$ depending on d_0 only such that $T_i(D_0)$ contains the annulus $B_1(0) \setminus B_{1-\delta_0}[0]$. Since $0 \notin S_i(D_1)$, f_i satisfies the hypothesis of the previous Claim, hence we may find $\delta_1 > 0$ depending on k and d_0 only such that $B_1(0) \setminus B_{1-\delta_1}[0] \subset S_i(D_1)$ and this implies that there exists a constant $C_4 > 0$ depending on k and d_0 only such that

$$\delta_i \geqslant C_4 s_i \qquad \text{for any} \quad i = 1, \ldots, n. \tag{A8}$$

Let us remark that, by (A4), we have, for any $i = 1, \ldots, n$,

$$0 < C_1/C_3 \leqslant \operatorname{cap}\left(B_{s_i}[w_i], B_1(0) \Big\backslash \bigcup_{j \neq i} B_{s_j}[w_j] \right) \leqslant \operatorname{cap}(B_{s_i}[w_i], B_{s_i+\delta_i}(w_i));$$

hence, using (A1), we deduce that there exists a constant $C_5 > 0$ depending on d_0 and k only such that

$$\delta_i \leqslant C_5 s_i \qquad \text{for any} \quad i = 1, \ldots, n. \tag{A9}$$

For any s_0, $0 < s_0 < 1$, let us split the interval $(0, s_0]$ into the subintervals $(s_0^{2^l}, s_0^{2^{l-1}}]$, $l = 1, 2, \ldots$. Due to (A2), the bound on the number of connected components of the multiple cavity of D_1, there exists $l \leqslant N+1$ such that $s_i \notin (s_0^{2^l}, s_0^{2^{l-1}})$ for every i. Hence there exists s, $0 < s \leqslant s_0$, depending on d_0 and s_0 only, such that if we set

$$I_1 = \{ i \in I : s_i \leqslant s^2 \}, \qquad I_2 = \{ i \in I : s_i \geqslant s \},$$

then $I = I_1 \cup I_2$.

Let us show $I_1 = \varnothing$ when s_0 is sufficiently small. By contradiction let us assume $I_1 \neq \varnothing$.

We take the condenser $(\bigcup_{i \in I_1} B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j])$ and we want to estimate its capacity. By subadditivity of capacity we have

$$\begin{aligned} &\operatorname{cap}\left(\bigcup_{i \in I_1} B_{s_i}[w_i], B_1(0) \Big\backslash \bigcup_{j \in I_2} B_{s_j}[w_j] \right) \\ &\qquad \leqslant \sum_{i \in I_1} \operatorname{cap}\left(B_{s_i}[w_i], B_1(0) \Big\backslash \bigcup_{j \in I_2} B_{s_j}[w_j] \right). \end{aligned} \tag{A10}$$

So let us fix $i \in I_1$ and let us evaluate $\operatorname{cap}(B_{s_i}[w_i], B_1(0) \setminus \bigcup_{j \in I_2} B_{s_j}[w_j])$. Assuming without loss of generality $s_0 \leqslant d_2$, by (A7) and (A8) applied to any $B_{s_j}[w_j]$ with $j \in I_2$, we have that

$$\operatorname{dist}\left(B_{s_i}[w_i], \bigcup_{j \in I_2} B_{s_j}[w_j] \cup \partial B_1(0) \right) \geqslant C_6 s \quad \text{for any} \quad i \in I_1, \tag{A11}$$

where C_6 depends on d_0 and on k only. Then

$$\operatorname{cap}\left(B_{s_i}[w_i], B_1(0)\Big\backslash \bigcup_{j\in I_2} B_{s_j}[w_j]\right)$$
$$\leqslant \operatorname{cap}(B_{s_i}[w_i], B_{s_i+C_6 s}(w_i)) \qquad \text{for any} \quad i \in I_1.$$

By (A1), since $s_i \leqslant s^2$,

$$\operatorname{cap}(B_{s_i}[w_i], B_{s_i+C_6 s}(w_i)) = 2\pi\left(\log\frac{s_i + C_6 s}{s_i}\right)^{-1}$$

$$\text{(A12)} \qquad \leqslant 2\pi\left(\log\frac{C_6}{s}\right)^{-1} \leqslant 2\pi\left(\log\frac{C_6}{s_0}\right)^{-1}.$$

Let us pick s_0 depending on k and d_0 only such that

$$\text{(A13)} \qquad 2\pi\left(\log\frac{C_6}{s_0}\right)^{-1} \leqslant C_1/(2C_3 N).$$

Then the combination of (A2), (A10), (A12) and (A13) violates the lower bound in (A4).

Hence we have found a positive constant s depending on k and d_0 only such that the minimal radius of the multiple cavity of D_1 is greater than s. Then, again by (A7) and (A8), also the separation distance may be bounded from below by a positive constant s depending on k and d_0 only. It remains to prove the Hölder continuity of χ and χ^{-1}. Given the bounds on the minimal radius and the separation distance of the multiple cavities of D_0 and D_1 respectively, this may be obtained by standard reflection arguments, see [18], with the help of our Claim to control the reflection around $\partial B_1(0)$. ∎

ACKNOWLEDGMENT

The authors thank Professor Masahiro Yamamoto for stimulating conversations on the subject of this work.

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